Introduction

Friday, April 16, 2021 8:38 AM

Reference: Section 10.4 of Lazarsfeld's "Positivity in Algebraic Geometry II"

Conjecture: (Fujita's conjecture)

X = smooth proj. variety of dim. n

A = ample divisor on X

Then: (i) If $m \geq n+1$, $K_X + mA$ is globally generated.

(ii) If $m \geq n+2$, $K_X + mA$ is very ample.

Remark: ① Cone theorem $\Rightarrow$ In (i), $K_X + mA$ is nef [$\overline{NE}(X) = \overline{NE}_{K_X \geq 0} + \sum \mathbb{R}_+[C_i]$ with $0 \leq -K_X \cdot C_i \leq n+1$]

In ii, $K_X + mA$ is ample [nef + ample = ample]

② $X = \mathbb{P}^n$, A = Hyperplane $\Big\}$ shows these bounds are sharp. [as $K_{\mathbb{P}^n} = -(n+1)H$]

Theorem: (Angehrn - Siu)

X = Smooth proj. variety of d

L = ample divisor on X

$x \in X$ closed point.

Assume that:

$$(L|_Z)^{\dim Z} > \binom{n+1}{2}^{\dim Z}$$

for every subvariety Z passing through x.

Then $K_X + L$ is free at x.

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[L ample $\therefore$ L^n = Growth of global sections of L

$H^0(\mathcal{O}_X(kL)) \approx \frac{k^n}{n!} \cdot L^n$ + lower order terms in k]

Remark: Theorem gives us quadratic bound for part (i) of Fujita's conjecture

Take $L = \left(\binom{n+1}{2} + 1\right) A$.

Theorem $\Rightarrow$ $K_X + \underbrace{\left(\binom{n+1}{2} + 1\right)}_{\approx n^2} A$ is globally generated.

Lemmas used in the proof

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Lemma: (Nadel vanishing)

X = smooth proj. variety

D = Effective $\mathbb{Q}$-divisor on X

L = Divisor s.t. $L-D$ is big and nef

Then:

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(D)) = 0 \qquad \forall\, i \geq 1$$

Lemma: (Constructing singular divisors)

- V = proj. variety of dim d.
- L = Ample divisor on V.
- $\alpha \in \mathbb{Q}_+$.
- $x \in V$ smooth point.

Assume that:

$$L^d > \alpha^d$$

Then $\exists$ an effective $\mathbb{Q}$-divisor D s.t. $D \equiv L$ and

$$\mathrm{mult}_x D > \alpha$$

Proof:

- Global sections of $kL = \dfrac{L^d}{d!} k^d$ + lower order terms
- Sections of kL vanishing up to order $k\alpha$ $= \binom{d + k\alpha - 1}{d}$

$$= \frac{\alpha^d}{d!} k^d + \text{Lower order terms}$$

As $L^d > \alpha^d$, for some large k, $\exists\, A \in |kL|$ which vanishes up to order $k\alpha$.

Finish by taking $D = \frac{1}{k} \cdot A$. $\blacksquare$

Remark: If x were a singular point, then same proof gives:

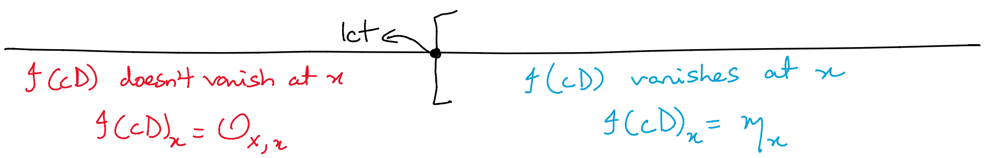
$$\operatorname{mult}_x D > \frac{\alpha}{e(\mathcal{O}_{X,x})^{1/d}} \qquad e(\mathcal{O}_{X,x}) = \text{Hilbert Samuel mult. of } X \text{ at } x$$

Defn: X = smooth variety

D = Effective $\mathbb{Q}$-divisor on X

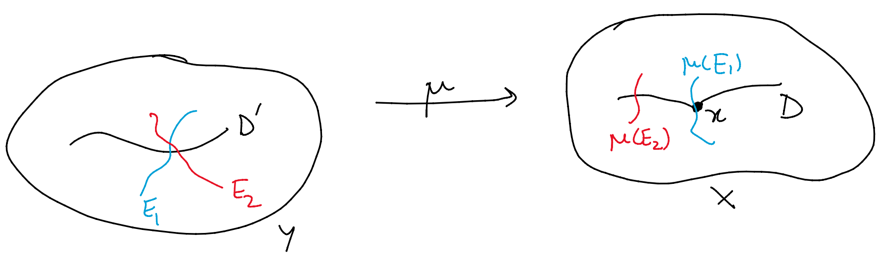
- The log canonical threshold of D at x is:

$$\operatorname{lct}(D;x) = \text{Smallest } c \text{ s.t. } \mathcal{J}(cD) \text{ is non-trivial at } x$$



- The log canonical locus of D at x is:

$$LC(D;x) = \operatorname{Zeros}(\mathcal{J}(cD)) \quad \text{where } c = \operatorname{lct}(D;x)$$



Let $K_{Y/X} = \sum b_i E_i$, $\mu^* D = D' + \sum r_i E_i$

Then
$$\mathcal{J}(cD) = \mu_*(\mathcal{O}_Y(K_{Y/X} - \lfloor c\mu^* D \rfloor))$$
$$= \mu_*(\mathcal{O}_Y(\underbrace{\textstyle\sum_i b_i - \lfloor c r_i \rfloor}))$$

This starts being non trivial at x once:

$$c = \min\left\{ \frac{b_i + 1}{r_i} \;\middle|\; \mu(E_i) \text{ passes through } x \right\}$$

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This is exactly $= \mathrm{lct}(D; x)$

Additionally, $LC(D;x) = \bigcup\limits_{\substack{\text{min. is} \\ \text{attained} \\ \text{for } E_i}} \mu(E_i)$

Lemma: $LC(D;x)$ can be made irreducible at x by adding a small effective divisor to D.

[$\exists\, E$ s.t. $LC(D+tE;x)$ is irred. at x $\forall\, 0 < t \ll 1$

Further, if L is an ample divisor on X, can choose $E = L$]

Proof idea: Choose E s.t. E contains only one component $\mu(E_{i_0})$.

E doesn't contain other components $\mu(E_i)$

Then $\min \left\{ \frac{b_i+1}{r_i} \right\}$ will be attained only at i_0 [as r_{i_0} is large]

$\therefore\ LC(D+tE;x) = \mu(E_{i_0})$ $\square$

Strategy of proof

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Theorem: (Angehrn - Siu)

X = smooth proj. variety of d

L = ample divisor on X

$x \in X$ closed point.

Assume that:

$$(L|_Z)^{\dim Z} > \binom{n+1}{2}^{\dim Z}$$

for every subvariety Z passing through x.

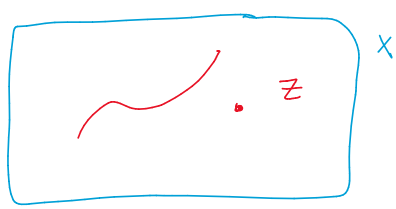
Then $K_X + L$ is free at x.

(i.e. $H^0(X, \mathcal{O}_X(K_X+L)) \longrightarrow H^0(x, \mathcal{O}_X(K_X+L)|_x)$ is surjective)

Strategy of proof

We will produce a divisor $D \equiv \lambda L$ for some $\lambda < 1$ s.t.

- $x \in Z := \text{Zeros}(\mathcal{J}(D))$
- x is an isolated point of Z



- Why is this enough?
  - $L - D \equiv L - \lambda L = (1-\lambda) L \rightarrow$ This is big and nef.

    $\therefore$ Nadel $\Rightarrow H^1(X, \mathcal{O}_X(K_X+L) \otimes \mathcal{J}(D)) = 0$

  - $\therefore$ $H^0(X, \mathcal{O}_X(K_X+L)) \longrightarrow H^0(Z, \mathcal{O}_X(K_X+L)|_Z)$ is surjective

    As $x \in Z$ isolated point, for any coherent sheaf F on Z:

    $$H^0(Z, F) \longrightarrow H^0(x, F|_x) \text{ is surjective}$$

    Thus $H^0(Z, \mathcal{O}_X(K_X+L)|_Z) \longrightarrow H^0(x, \mathcal{O}_X(K_X+L)|_x)$ is surjective.

Thus $H^0(Z, \mathcal{O}_X(K_X+L)|_Z) \longrightarrow H^0(x, \mathcal{O}_X(K_X+L)|_x)$ is surjective.

$\therefore$ $H^0(X, \mathcal{O}_X(K_X+L)) \longrightarrow H^0(x, \mathcal{O}_X(K_X+L)|_x)$ is surjective.

- How do we produce D?
  - First produce D_1 s.t. $D_1 \equiv \lambda_1 L$, $\lambda_1 < 1$

    $Z_1 := \text{Zeros}(\mathcal{J}(D_1))$ contains x

    $\dim(Z_1) \leq n-1$ about x

    Z_1 is irred. at x

  - Produce D_2 s.t. $D_2 \equiv \lambda_2 L$, $\lambda_2 < 1$

    $Z_2 := \text{Zeros}(\mathcal{J}(D_2))$ contains x

    $Z_2 \subsetneq Z_1$

    $[\Rightarrow \dim(Z_2) \leq n-2$ about $x]$

    Z_2 irred at x.

  - Irreducibility at each step $\Rightarrow$ After n steps, we end up with $D_n \equiv \lambda_n L$, $\lambda_n < 1$

    $Z_n := \text{Zeros}(\mathcal{J}(D_n))$ contains x

    $\dim(Z_n) = 0$ about x i.e. x is an isolated point of Z_n.

  D_n is the D we required.

First attempt of proof

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Theorem: (Angehrn - Siu)

$X =$ Smooth proj. variety of d

$L =$ ample divisor on X

$x \in X$ closed point.

Assume that:

$$(L|_Z)^{\dim Z} > \binom{n+1}{2}^{\dim Z}$$

for every subvariety Z passing through x.

Then $K_X + L$ is free at x.

(i.e. $H^0(X, \mathcal{O}_X(K_X+L)) \longrightarrow H^0(x, \mathcal{O}_X(K_X+L)|_x)$ is surjective)

Proof - First attempt

Step 1: Need to produce a $D_1 \equiv \lambda_1 L$ with λ_1 small s.t. $x \in Z_1 := \mathrm{Zeros}(\mathcal{J}(D_1))$

How?

Choose D_1 very singular at x:

$$\mathrm{mult}_x(D_1) > n \implies x \in Z_1 := \mathrm{Zeros}(\mathcal{J}(D_1)).$$

Observe that $L^n > \binom{n+1}{2}^n$

Constructing singular divisor lemma $\Rightarrow \exists\ D \equiv L$ s.t.

$$\mathrm{mult}_x D > \binom{n+1}{2}$$

Set $\tilde{D} = \frac{n}{\binom{n+1}{2}} D$. Thus $\mathrm{mult}_x \tilde{D} > n$.

- Observe $\mathrm{lct}(\tilde{D}; x) < 1$

  Make $\mathrm{lct} = 1$ by replacing $\tilde{D}$ with $D_1 = \mathrm{lct}(\tilde{D}; x) \cdot \tilde{D}$

  Thus $D_1 \equiv \mathrm{lct}(\tilde{D}; x) \cdot \frac{n}{\binom{n+1}{2}} \cdot L = \lambda_1 L$ where $\lambda_1 < \frac{n}{\binom{n+1}{2}}$

Thus $\quad D_1 \equiv \operatorname{lct}(\tilde{D}; x) \cdot \frac{n}{\binom{n+1}{2}} \cdot L = \lambda_1 L \quad$ where $\lambda_1 < \frac{n}{\binom{n+1}{2}}$

$$LC(D_1; x) = Z_1 := \text{Zeros}(\mathcal{J}(D_1))$$

Upshot: $\mathcal{J}((1-\varepsilon)D_1)$ vanishes at a strictly smaller subset of Z_1.

- Make LC locus irreducible by adding a small multiple of L.

  Normalize again to make $\operatorname{lct}(D_1; x) = 1$

  Thus, finally we'll get:

  $D_1 \equiv \lambda_1 L \quad$ with $\lambda_1 < \frac{n}{\binom{n+1}{2}}$

  $Z_1 := \text{Zeros}(\mathcal{J}(D_1))$ contains x

  Z_1 is irred. at x

  $\dim(Z_1) \leq n-1$ about x

  $\text{Zeros}(\mathcal{J}((1-\varepsilon)D_1))$ is strictly smaller, doesn't contain x.

Assumption: For ease of exposition, say $\dim Z_1 = n-1$.

Warning: There will be a mistake in the proof below. Please watch out for it and point it out!

Step 2: Repeat the previous step on Z_1 i.e.:

- Find a divisor $\bar{B}$ on Z_1 s.t. $\bar{B}$ is very singular at x

  $\text{mult}_x(\bar{B}) > n-1 \implies x \in \text{Zeros}(\mathcal{J}(\bar{B}))$

  How?

  Hypothesis of Angehrn - Siu $\implies \bar{L}^{n-1} > \binom{n+1}{2}^{n-1}$

  Constructing singular divisor lemma $\implies \exists\, \bar{D} \equiv \bar{L}$ s.t.

  $$\text{mult}_x \bar{D} > \binom{n+1}{2}$$

  Set $\bar{B}$ [illegible] $\bar{D}$ Thus mult $\bar{B} > n-1$

$$\mathrm{mult}_x D > \binom{n+1}{2}$$

Set $\overline{B} = \dfrac{n-1}{\binom{n+1}{2}} \overline{D}$. Thus $\mathrm{mult}_x \overline{B} > n-1$.

Now, let B be a lift of $\overline{B}$.

Then look at $D_2 = (1-\varepsilon) D_1 + B$.

Claim: The lift B can be carefully chosen so that:

$Z_2 := \mathrm{Zeros}(\mathcal{J}(D_2)) \subsetneq Z_1$

$x \in Z_2$.

Proof: Slightly technical. Here's a rough heuristic.

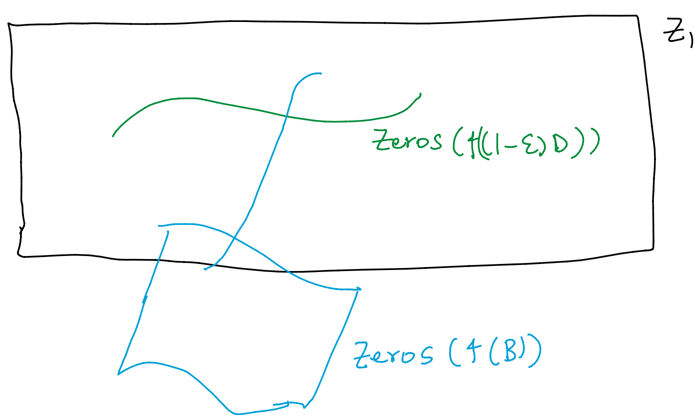
2 things to show: (i) $Z_2 \subseteq Z_1$

(ii) Containment is strict

We know $\mathrm{Zeros}(\mathcal{J}((1-\varepsilon)D_1))$ is strictly contained)

As B doesn't contain Z_1, adding B still gives us that

$\mathrm{Zeros}(\mathcal{J}((1-\varepsilon)D_1 + B)) \subsetneq Z_1$



Some of this could lie outside $\blacksquare$

Thus we got a D_2 s.t. $D_2 = (1-\varepsilon) D_1 + B$

$\Rightarrow D_2 \equiv \lambda_2 L$ where $\lambda_2 < \dfrac{n}{\binom{n+1}{2}} + \dfrac{n-1}{\binom{n+1}{2}}$

$Z_2 := \mathrm{Zeros}(\mathcal{J}(D_2))$ contains x

Z_2 is irred. at x

$Z_2 := \text{zeros}(\mathcal{J}(D_2))$ contains x

Z_2 is irred. at x

$\dim(Z_2) \leq n-2$ about x

We can now repeat this n times to finish the proof.

$$D_n \equiv \lambda L \quad \text{where} \quad \lambda < \frac{n}{\binom{n+1}{2}} + \frac{n-1}{\binom{n+1}{2}} + \frac{n-2}{\binom{n+1}{2}} + \dots + \frac{1}{\binom{n+1}{2}}$$

$$\Rightarrow \lambda < 1$$

Question: What is the mistake?

Answer: x could be a highly singular point of Z_1.

Thus we can't find $\bar{B} \equiv$ small multiple of $\bar{L}$ s.t.:

$\bar{B}$ is very singular at x

Fixing the proof

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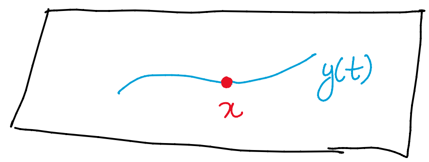
Problem: In Step 2, x need not be a smooth point of Z_1.

Angehrn-Siu fix the proof as follows:

- Choose a curve $y(t)$ in Z_1 so that $y(0) = x$

  $y(t)$ = Smooth point of Z_1 $\forall t \neq 0$



- Construct a family of divisors $\overline{B}_t$ on Z_1 $\forall t \neq 0$ s.t.:

  $\overline{B}_t$ is a small multiple of $\overline{L}$.

  $\overline{B}_t$ has high multiplicity at $y(t)$.

  [Can do this because $y(t)$ = Smooth point of Z_1]

  $\therefore$ $\overline{B}_t \equiv \frac{n-1}{\binom{n+1}{2}} \overline{L}$ and $\operatorname{mult}_{y(t)} B_t > n-1$

  Lifting these $\overline{B}_t$ appropriately to get B_t.

  $y(t) \in \text{Zeros}\,(\mathcal{J}((1-\varepsilon)D_1 + B_t))$

- Define the limit of these divisors to be B_0.

  By semicontinuity of log discrepancies:

  $x = y(0) \in \text{Zeros}\,(\mathcal{J}((1-\varepsilon)D_1 + B_0))$

  Thus B_0 is the required B.

[This whole fix is the heart of the proof.
Gets quite technical.
"Cutting down LC locus lemma" - Prop 10.4.10 in Lazarsfeld]

$\blacksquare$

QUESTIONS?